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On the most general boundary conditions for the Aharonov–Bohm scattering of a Dirac particle: helicity and Aharonov–Bohm symmetry conservation

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Abstract

We show that the Hamiltonian H and the helicity operator Λ of a Dirac particle moving in two dimensions in the presence of an infinitely thin magnetic flux tube each admit a four-parameter family of self-adjoint extensions. Each extension is in one-to-one correspondence with the boundary conditions (BCs) to be satisfied by the eigenfunctions at the origin. Although the actions of these two operators commute before specification of BCs, to ensure helicity conservation it is not sufficient to take the same BCs for both operators. We show that, given certain relations between the parameters of the extensions, it is possible to write down the most general domain where both operators H and Λ are self-adjoint with helicity conservation and also Aharonov–Bohm symmetry ($\phi \to \phi + 1$) preserved, where ϕ is the magnetic flux in natural units. The continuity of the dynamics is also obtained. Our results imply that neither helicity conservation nor Aharonov–Bohm symmetry by itself solves the problem of choosing the 'physical' BCs for this system.

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1. Introduction

In a previous article [1] we considered the Hamiltonian operator H of a Dirac particle of mass $m > 0$, moving in two dimensions in the presence of an infinitely thin magnetic flux tube at the origin, formally defined as

$$H = \left[\vec{p} + \frac{e}{c}\vec{A}\right] \cdot \vec{\alpha} + \beta m \qquad (1.1)$$

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where $\vec{p} = (p_x, p_y)$, $\vec{\alpha} = (\alpha_1, \alpha_2)$

$$\begin{aligned} \alpha_i &= \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \qquad i = 1, 2 \\ \beta &= \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \end{aligned} \tag{1.2}$$

where σ_1, σ_2 and σ_3 are the Pauli matrices. The vector potential, in the Coulomb gauge, is

$$\frac{e}{c}\vec{A} = \frac{\phi}{r^2}(-y, x)\,. \tag{1.3}$$

We considered also the helicity operator given as follows:

$$\Lambda = \left[\vec{p} + \frac{e}{c}\vec{A}\right] \cdot \vec{\Sigma} \tag{1.4}$$

where $\vec{\Sigma} = (\Sigma_1, \Sigma_2)$

$$\Sigma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \qquad i = 1, 2. \tag{1.5}$$

These two operators H and Λ formally commute as one can easily verify. However this is not sufficient to guarantee common eigenfunctions and hence, as we shall see, helicity conservation [1]. To become self-adjoint operators, both operators H and Λ require specification of boundary conditions (BCs) at the origin to be satisfied by their eigenfunctions as first shown by de Sousa Gebert [2]. Surprisingly helicity conservation is not obtained by only taking the same BCs for H and Λ, i.e. only by defining the same domain for both operators as claimed in [1].

Remark 1.1. Two operators are said to commute if and only if all the projections in their associated projection-valued measures commute (see [7], p 271). This is equivalent to having a complete set of common generalized eigenfunctions (see [9]), known to physicists as eigenfunctions in the continuous.

In fact, it is not possible to have helicity conservation for all values of the parameters of the class of admissible BCs considered in [1]. As we are going to show in this paper, this occurs only for certain values of the parameters of the extensions of H and Λ obtained in [1]. This fact occurs because both operators H and Λ, when acting in a common domain, do not leave it invariant, as we demonstrate. This problem may have some analogy with the Nelson problem of [7].

As shown in [1] and repeated here for completeness, the operators H and Λ each admit a four-parameter family of self-adjoint realizations in one-to-one correspondence with the BCs to be satisfied by the eigenfunctions at the origin. Given certain relations between the parameters of the extensions, we can have both operators defined in the same domain with helicity conservation and also with preservation of Aharonov–Bohm symmetry ($\phi \to \phi + 1$).

This paper is organized as follows. In section 2 we write down the most general eigenfunctions of each operator H and Λ and obtain a formal condition for the operators to have common eigenfunctions. In section 3 we describe the most general admissible BCs for the Hamiltonian operator parametrized by a (2×2) unitary matrix with four parameters. In section 4 we analyse the possibility that the formal condition obtained in section 2 is satisfied by the eigenfunctions of H in the domain obtained in section 3. This results in some relations between the parameters, so we analyse whether the helicity operator is self-adjoint in the domain defined in section 3 with the extensions' parameters satisfying the relations obtained. We verify that indeed the requirement of self-adjointness for Λ in that domain does not lead to any extra relation between the parameters.

In section 5 we show that the Aharonov–Bohm symmetry is preserved for the dynamics given by the BCs found. In section 6 we present our conclusions.

Finally, in the appendix we return to the situation discussed in [1], which is a particular case of the BCs obtained in section 2. We show that the helicity and the Hamiltonian operators do not leave the domain invariant. So in this case it does not make any sense to talk about commutativity between these operators and consequently about common eigenfunctions. We show that only for some particular values of the parameters of the extensions, that result in the same BCs as [3] and [4], is it possible to have helicity conservation for the dynamics given by H.

Our results imply that, for the problem of a Dirac particle moving in two dimensions in the presence of an infinitely thin tube of magnetic flux, neither helicity conservation nor the Aharonov–Bohm symmetry by itself solve the problem of choosing the 'physical' boundary condition. This may be important with respect to the problem of Aharonov–Bohm interaction of cosmic strings with matter [5].

2. Most general eigenfunctions of the operators H and Λ—formal condition for commutativity

In polar coordinates (r, φ), after separation of variables

$$\psi_\pm(r,\varphi) = \begin{pmatrix} \psi^1_\pm(r) \\ \psi^2_\pm(r)\,\mathrm{e}^{\mathrm{i}\varphi} \\ \psi^3_\pm(r) \\ \psi^4_\pm(r)\,\mathrm{e}^{\mathrm{i}\varphi} \end{pmatrix} \mathrm{e}^{\mathrm{i}n\varphi} \qquad n \in Z \tag{2.1}$$

the eigenvalue equation for H reads

$$H(r)\psi_\pm(r) = \begin{bmatrix} h+\sigma_3 m & 0 \\ 0 & h-\sigma_3 m \end{bmatrix} \psi_\pm(r) = \pm|E|\,\psi_\pm(r) \tag{2.2}$$

where

$$h = \begin{bmatrix} 0 & -\mathrm{i}\left(\partial_r + \frac{\nu+1}{r}\right) \\ -\mathrm{i}\left(\partial_r - \frac{\nu}{r}\right) & 0 \end{bmatrix} \tag{2.3}$$

and $\nu = n + \phi$ and where we define

$$\psi_\pm(r) = \begin{pmatrix} \psi^1_\pm(r) \\ \psi^2_\pm(r) \\ \psi^3_\pm(r) \\ \psi^4_\pm(r) \end{pmatrix} = \begin{pmatrix} \phi^1_\pm(r) \\ \phi^2_\pm(r) \end{pmatrix} \tag{2.4}$$

where $\phi^1_\pm$ and $\phi^2_\pm$ are two-component spinors.

Note that the Hamiltonian operator of equation (2.2) above has a symmetry $\phi \to \phi + 1$; this is the Aharonov–Bohm symmetry [3].

Both operators H and Λ commute with the total angular momentum operator

$$J_z = \frac{\Sigma_3}{2} + l_z$$

where

$$\Sigma_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \tag{2.5}$$

and

$$l_z = xp_y - yp_x \tag{2.6}$$

and we can also easily verify that the eigenfunctions of equation (2.1) have $j_z = n + \frac{1}{2}$.

As the Hamiltonian (1.1) and (2.2) decomposes into two uncoupled operators acting on two-component spinors, the eigenvalue problem given by (2.2) is equivalent to the following problem:

$$(h + m\sigma_3)\,\phi^1_\pm = \pm\,|E|\,\phi^1_\pm \tag{2.7}$$

$$(h - m\sigma_3)\,\phi^2_\pm = \pm\,|E|\,\phi^2_\pm. \tag{2.8}$$

Writing the two-component spinor $\phi^i_\pm$ as

$$\phi^j_\pm = \begin{pmatrix} u^j_\pm \\ v^j_\pm \end{pmatrix} \tag{2.9}$$

equations (2.7) and (2.8) imply

$$\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{\nu^2}{r^2} + k^2\right) u^j_\pm = 0 \tag{2.10}$$

where $k = \sqrt{E^2 - m^2}$ and also lead to the important relation between the upper ($u_\pm$) and lower ($v_\pm$) components

$$v^j_\pm = \frac{-\mathrm{i}}{\pm\,|E| + ms}\left(\partial_r - \frac{\nu}{r}\right) u^j_\pm \tag{2.11}$$

where $s = +1$ ($s = -1$) corresponds to $\phi^1_\pm$ ($\phi^2_\pm$).

Solving (2.10) and using (2.11) we obtain

$$\phi^1_\pm(kr) = \begin{pmatrix} J_{-\nu}(kr) + c_\pm J_\nu(kr) \\ \frac{-\mathrm{i}k}{\pm|E|+m}\left(J_{-\nu-1}(kr) - c_\pm J_{\nu+1}(kr)\right) \end{pmatrix} \tag{2.12}$$

$$\phi^2_\pm(kr) = \begin{pmatrix} J_{-\nu}(kr) + d_\pm J_\nu(kr) \\ \frac{-\mathrm{i}k}{\pm|E|-m}\left(J_{-\nu-1}(kr) - d_\pm J_{\nu+1}(kr)\right) \end{pmatrix} \tag{2.13}$$

where $c_\pm$ and $d_\pm$ are two arbitrary constants.

So, the most general eigenfunctions for H can be written as follows:

$$\psi_\pm(kr) = \begin{pmatrix} \alpha\phi^1_\pm(kr) \\ \beta\phi^2_\pm(kr) \end{pmatrix} \tag{2.14}$$

where α and β are arbitrary constants. For a fixed value of the energy, the eigenfunctions are a proportion of the lower ($\phi^2_\pm$) and upper ($\phi^1_\pm$) components.

In polar coordinates, after the same separation of variables given by (2.1), the eigenvalue equation for Λ reads

$$\Lambda(r)\chi_\pm(r) = \begin{bmatrix} 0 & h \\ h & 0 \end{bmatrix} \chi_\pm(r) = \pm\,|\lambda|\,\chi_\pm(r)\,. \tag{2.15}$$

This problem can be solved in a similar way as for the Hamiltonian operator, leading to two independent solutions $\chi'_\pm$ and $\chi''_\pm$ given by

$$\chi'_\pm(\lambda r) = \begin{pmatrix} \Phi(\lambda r) \\ \pm\Phi'(\lambda r) \end{pmatrix} \tag{2.16}$$

and

$$\chi''_\pm(\lambda r) = \begin{pmatrix} \pm\Phi'(\lambda r) \\ \Phi(\lambda r) \end{pmatrix} \tag{2.17}$$

where

$$\Phi(\lambda r) = \begin{pmatrix} J_{-\nu}(\lambda r) + aJ_\nu(\lambda r) \\ J_{-\nu-1}(\lambda r) + bJ_{\nu+1}(\lambda r) \end{pmatrix} \tag{2.18}$$

and

$$\Phi'(\lambda r) = \begin{pmatrix} -\mathrm{i}\,(bJ_\nu(\lambda r) - J_{-\nu}(\lambda r)) \\ -\mathrm{i}\,(J_{-\nu-1}(\lambda r) - aJ_{\nu+1}(\lambda r)) \end{pmatrix} \tag{2.19}$$

where a and b are two arbitrary constants.

We are going to impose that the most general eigenfunctions of the Hamiltonian given by (2.14) should also be eigenfunctions of helicity given by (2.15)

$$\begin{aligned} &\Lambda(r)\psi_\pm(kr) = \lambda\psi_\pm(kr) \\ &\begin{bmatrix} 0 & h \\ h & 0 \end{bmatrix} \begin{pmatrix} \alpha\phi^1_\pm(kr) \\ \beta\phi^2_\pm(kr) \end{pmatrix} = \begin{pmatrix} \beta h\phi^2_\pm \\ \alpha h\phi^1_\pm \end{pmatrix} = \lambda \begin{pmatrix} \alpha\phi^1_\pm(kr) \\ \beta\phi^2_\pm(kr) \end{pmatrix}. \end{aligned} \tag{2.20}$$

Using equations (2.7) and (2.8), the upper component of equation (2.20) leads to the system

$$\begin{aligned} &\beta(\pm|E| + m) = \lambda\alpha \\ &\beta(\pm|E| + m)\,d_\pm = \lambda\alpha c_\pm \end{aligned} \tag{2.21}$$

which has a non-trivial solution under the condition

$$c_\pm = d_\pm. \tag{2.22}$$

The lower component of equation (2.20) leads to the system

$$\begin{aligned} &\alpha(\pm|E| - m) = \lambda\beta \\ &\alpha(\pm|E| - m)\,c_\pm = \lambda\beta d_\pm \end{aligned} \tag{2.23}$$

which has a non-trivial solution under the same condition of equation (2.22).

Thus, under condition (2.22) the two systems above can be further reduced to

$$\begin{aligned} &\beta(\pm|E| + m) = \lambda\alpha \\ &\alpha(\pm|E| - m) = \lambda\beta. \end{aligned} \tag{2.24}$$

Notice that the system above has a solution if and only if

$$\lambda = \pm\sqrt{E^2 - m^2} = \pm k. \tag{2.25}$$

For $\lambda = +k$, we have

$$\beta = \frac{k}{\pm|E| + m}\alpha. \tag{2.26}$$

For $\lambda = -k$, we have

$$\beta = \frac{-k}{\pm|E| + m}\alpha. \tag{2.27}$$

Thus, taking α and β as in equations (2.26) and (2.27) and with condition (2.22) satisfied in a common domain of H and Λ we can certainly have common eigenfunctions.

Remark 2.1. Note that the Hamiltonian of equation (2.2) has one bound state of the form of the bound state obtained in [2]:

$$\psi(r) = \begin{pmatrix} K_\nu(kr) \\ \mathrm{i}\sqrt{\frac{m-E}{m+E}}K_{\nu+1}(kr) \\ K_\nu(kr) \\ \mathrm{i}\sqrt{\frac{m+E}{m-E}}K_{\nu+1}(kr) \end{pmatrix}$$

for E in the gap $-m \leqslant E \leqslant m$. The helicity operator does not even have formal bound states, so the existence of a bound state would destroy commutativity. However, as we shall see, the Hamiltonian operator has no bound state in the common domain given by the BCs that we find in section 4.

3. The most general admissible boundary conditions

To become self-adjoint operators, the Hamiltonian operator of equation (2.2) and the helicity operator of equation (2.15) require specification of BCs to be satisfied by the eigenfunctions at the origin when $\nu = n + \phi$ is in the open interval $(-1, 0)$. It is impossible to take the usual regularity assumption for both (lower and upper) spinor components ([1, 2]).

The problem of determining the admissible BCs is identical to the problem of determining the SA extensions of a densely defined symmetric operator. We construct the SA extensions of H and Λ by the method of deficiency indices developed by von Neumann. This theory requires first the determination of the deficiency spaces of the two operators when acting in a common dense domain D_0 of square integrable functions that are regular at the origin, so we start with H and Λ defined in the domain $C_0^\infty\left(R^2\backslash\{0\}\right)^4$.

The deficiency spaces $D_\pm(H)$, $D_\pm(\Lambda)$ are defined by

$$H^*\varphi_\pm = \pm\mathrm{i}m\varphi_\pm \qquad \text{if} \quad \varphi_\pm \in D_\pm(H) \tag{3.1}$$

$$\Lambda^*\chi_\pm = \pm\mathrm{i}l\chi_\pm \qquad \text{if} \quad \chi_\pm \in D_\pm(\Lambda) \tag{3.2}$$

where H^* and Λ^* denote the adjoint operators of H and Λ respectively. We insert m and l for dimensional reasons.

Since the four-component equation (3.1) decouples into two two-component equations [1, 3], we can easily verify that the subspaces $D_\pm(H)$ are generated by

$$\varphi_\pm^1 = \begin{bmatrix} \eta_\pm^1 \\ 0 \end{bmatrix} \tag{3.3}$$

$$\varphi_\pm^2 = \begin{bmatrix} 0 \\ \eta_\pm^2 \end{bmatrix} \tag{3.4}$$

where the two-component spinors $\eta_\pm^{1,2}$ are given by

$$\eta_\pm^1 = \frac{1}{N}\begin{bmatrix} K_\nu(\sqrt{2}mr) \\ \pm\mathrm{e}^{\pm\mathrm{i}\frac{\pi}{4}}K_{\nu+1}(\sqrt{2}mr) \end{bmatrix} \tag{3.5}$$

and

$$\eta_\pm^2 = \frac{1}{N}\begin{bmatrix} K_\nu(\sqrt{2}mr) \\ \pm\mathrm{e}^{\mp\mathrm{i}\frac{\pi}{4}}K_{\nu+1}(\sqrt{2}mr) \end{bmatrix} \tag{3.6}$$

where N is an overall normalization constant. If we look at H as a pair of uncoupled operators acting on a two-component spinor, we obtain the deficiency indices $d_\pm(H) = 1$ for each operator of the pair.

There is a unitary equivalence that connects the two formal operators (1.1) and (1.4) [1]

$$\Lambda = U(H - \beta m)U^{-1} \tag{3.7}$$

where

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & -\mathrm{i}\sigma_3 \\ I & \mathrm{i}\sigma_3 \end{bmatrix}. \tag{3.8}$$

Noting this equivalence the problem of BCs for Λ can be read from the corresponding problem for $(H - \beta m)$.

In an analogous way, we obtain the deficiency subspaces of $(H - \beta m)$ by solving

$$(H - \beta m)^*\xi_\pm = \pm\mathrm{i}l\xi_\pm. \tag{3.9}$$

The deficiency subspaces $D_{\pm}(H - \beta m)$ are generated by the normalized functions

$$\xi_{\pm}^{1} = \begin{bmatrix} \Phi_{\pm} \\ 0 \end{bmatrix} \tag{3.10}$$

$$\xi_{\pm}^{2} = \begin{bmatrix} 0 \\ \Phi_{\pm} \end{bmatrix} \tag{3.11}$$

where

$$\Phi_{\pm} = \frac{1}{N} \begin{bmatrix} K_{\nu}(lr) \\ \pm K_{\nu+1}(lr) \end{bmatrix} \tag{3.12}$$

where N is a normalization constant.

Therefore, we obtain the deficiency indices $d_{\pm}(H - \beta m) = 2$, and also the deficiency indices $d_{\pm}(\Lambda) = 2$, because of the unitary equivalence given by equation (3.7).

The admissible BCs for the vector ψ, in the domain of the self-adjoint extensions of a symmetric operator A with equal deficiency indices $d_{+}(A) = d_{-}(A) = n$, are parametrized by an $(n \times n)$ matrix W through the equations ([1, 8])

$$\left\langle A^{*}\left[f_{+}^{j} + \sum_{k=1}^{n} w_{jk} f_{-}^{k} \right], \psi \right\rangle = \left\langle \left[f_{+}^{j} + \sum_{k=1}^{n} w_{jk} f_{-}^{k} \right], A^{*}\psi \right\rangle \tag{3.13}$$

where $f_{\pm}^{j}$ are the normalized and mutually orthogonal vectors in $D_{\pm}(A)$, such that

$$A^{*} f_{\pm}^{j} = \pm \mathrm{i} a f_{\pm}^{j} \qquad j = 1, \ldots, n. \tag{3.14}$$

So the BCs for H can be parametrized by a (2×2) unitary matrix

$$W = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \tag{3.15}$$

through the equations

$$\lim_{r \to 0} r \left[\varphi_{+}^{1}(r) + w_{11} \varphi_{-}^{1}(r) + w_{12} \varphi_{-}^{2}(r) \right]^{\dagger} \alpha_{1} \psi(r) = 0 \tag{3.16}$$

$$\lim_{r \to 0} r \left[\varphi_{+}^{2}(r) + w_{21} \varphi_{-}^{1}(r) + w_{22} \varphi_{-}^{2}(r) \right]^{\dagger} \alpha_{1} \psi(r) = 0. \tag{3.17}$$

In an analogous way, the admissible BCs for $H - \beta m$ can be parametrized by a (2×2) unitary matrix V

$$V = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \tag{3.18}$$

through the equations

$$\lim_{r \to 0} r \left[\chi_{+}^{1}(r) + v_{11} \chi_{-}^{1}(r) + v_{12} \chi_{-}^{2}(r) \right]^{\dagger} \alpha_{1} \psi(r) = 0 \tag{3.19}$$

$$\lim_{r \to 0} r \left[\chi_{+}^{2}(r) + v_{21} \chi_{-}^{1}(r) + v_{22} \chi_{-}^{2}(r) \right]^{\dagger} \alpha_{1} \psi(r) = 0. \tag{3.20}$$

Using the unitary equivalence (3.7) the admissible BCs for the Λ operator can be written as

$$\lim_{r \to 0} r \left[U \chi_{+}^{1}(r) + U v_{11} \chi_{-}^{1}(r) + v_{12} U \chi_{-}^{2}(r) \right]^{\dagger} \Sigma_{1} U \psi(r) = 0 \tag{3.21}$$

$$\lim_{r \to 0} r \left[U \chi_{+}^{2}(r) + v_{21} U \chi_{-}^{1}(r) + v_{22} U \chi_{-}^{2}(r) \right]^{\dagger} \Sigma_{1} U \psi(r) = 0 \tag{3.22}$$

where we have used the unitarity of U and also

$$U \alpha_{1} U^{-1} = \Sigma_{1}. \tag{3.23}$$

The equations (3.16) and (3.17) give all the self-adjoint extensions of H, parametrized by a four-parameter family in one-to-one correspondence with the BCs to be satisfied by the eigenfunctions at the origin.

In an analogous way, equations (3.21) and (3.22) give all the self-adjoint extensions for Λ also parametrized by a four-parameter family, which exhaust the class of admissible BCs for the helicity operator.

We are going to write the unitary matrix of equation (3.15) as

$$W = \begin{pmatrix} \cos\theta \mathrm{e}^{\mathrm{i}(\varphi+\alpha)} & \mathrm{i}\sin\theta \mathrm{e}^{-\mathrm{i}(\psi-\alpha)} \\ \mathrm{i}\sin\theta \mathrm{e}^{\mathrm{i}(\psi+\alpha)} & \cos\theta \mathrm{e}^{-\mathrm{i}(\varphi-\alpha)} \end{pmatrix}. \tag{3.24}$$

Then, equations (3.16) and (3.17) become

$$\lim_{r\to 0} r\left[\varphi_+^1(r) + \cos\theta \mathrm{e}^{\mathrm{i}(\varphi+\alpha)}\varphi_-^1(r) + \mathrm{i}\sin\theta \mathrm{e}^{-\mathrm{i}(\psi-\alpha)}\varphi_-^2(r)\right]^\dagger \alpha_1 \psi(r) = 0 \tag{3.25}$$

$$\lim_{r\to 0} r\left[\varphi_+^2(r) + \mathrm{i}\sin\theta \mathrm{e}^{\mathrm{i}(\psi+\alpha)}\varphi_-^1(r) + \cos\theta \mathrm{e}^{-\mathrm{i}(\varphi-\alpha)}\varphi_-^2(r)\right]^\dagger \alpha_1 \psi(r) = 0. \tag{3.26}$$

Let us now introduce the quantities

$$x = \frac{1}{2}\Gamma(1+\nu)\left[\frac{m}{\sqrt{2}}\right]^{-(1+\nu)} \lim_{r\to 0} r^{-\nu}\psi^1(r) \tag{3.27}$$

$$y = \frac{1}{2}\Gamma(-\nu)\left[\frac{m}{\sqrt{2}}\right]^{\nu} \lim_{r\to 0} r^{1+\nu}\psi^2(r) \tag{3.28}$$

$$z = \frac{1}{2}\Gamma(1+\nu)\left[\frac{m}{\sqrt{2}}\right]^{-(1+\nu)} \lim_{r\to 0} r^{-\nu}\psi^3(r) \tag{3.29}$$

$$w = \frac{1}{2}\Gamma(-\nu)\left[\frac{m}{\sqrt{2}}\right]^{\nu} \lim_{r\to 0} r^{1+\nu}\psi^4(r). \tag{3.30}$$

Using the definitions above and substituting equations (3.3)–(3.6) in (3.25) and (3.26), we have

$$\begin{aligned} &\left(\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}} - \cos\theta \mathrm{e}^{-\mathrm{i}(\varphi-\alpha)}\right)x + \left(1 + \cos\theta \mathrm{e}^{-\mathrm{i}(\varphi+\alpha)}\right)y + \mathrm{i}\sin\theta \mathrm{e}^{-\mathrm{i}(\psi+\alpha)}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}z \\ &\qquad -\mathrm{i}\sin\theta \mathrm{e}^{-\mathrm{i}(\psi-\alpha)}w = 0 \end{aligned} \tag{3.31}$$

$$\begin{aligned} &\mathrm{i}\sin\theta \mathrm{e}^{-\mathrm{i}(\psi+\alpha)}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}x - \mathrm{i}\sin\theta \mathrm{e}^{-\mathrm{i}(\psi+\alpha)}y + \left(\mathrm{e}^{\mathrm{i}\frac{\pi}{4}} - \cos\theta \mathrm{e}^{-\mathrm{i}(\varphi-\alpha)}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\right)z \\ &\qquad +\left(1 + \cos\theta \mathrm{e}^{-\mathrm{i}(\varphi-\alpha)}\right)w = 0. \end{aligned} \tag{3.32}$$

The equations (3.31) and (3.32) above define the most general domain where the Hamiltonian is a self-adjoint operator and also include the BC presented in [1–5]. Notice that, since these BCs couple the four components of the wavefunctions, they do not preserve the original property of the Hamiltonian of being represented by two independent operators acting on two-component spinors.

4. Common eigenfunctions for H and Λ

As we have shown in section 2, it is formally possible to obtain the common eigenfunctions of H and Λ only for some particular linear combinations of equation (2.14) given by (2.26) and (2.27) with the condition (2.22) satisfied.

We are going to consider an eigenfunction of H, where α and β are given by equations (2.26) and (2.27).

Imposing that these eigenfunctions satisfy the boundary conditions (3.31) and (3.32) and also that the condition (2.22) is satisfied, we obtain the following relations for the parameters of the extensions:

$$\alpha = n_1\pi \qquad n_1 = 0, 1, 2, \ldots \tag{4.1}$$

$$\psi = n_2\pi \qquad n_2 = 0, 1, 2, \ldots \tag{4.2}$$

$$\cos\theta\,(\sin\varphi - \cos\varphi) = (-1)^{n_1} \tag{4.3}$$

$$\cos\theta\sin\varphi \neq (-1)^{n_1}. \tag{4.4}$$

In this situation, equation (2.22) reads

$$c_- = d_- = c_+ = d_+ = +\frac{\sqrt{2}}{2}\frac{(-1)^{n_1+n_2}\sin\theta}{(1-(-1)^{n_1}\cos\theta\sin\varphi)}\left(\frac{k}{\sqrt{2}m}\right)^{2|\nu|-1} \tag{4.5}$$

for $\lambda = +k$ and

$$c_- = d_- = c_+ = d_+ = -\frac{\sqrt{2}}{2}\frac{(-1)^{n_1+n_2}\sin\theta}{(1-(-1)^{n_1}\cos\theta\sin\varphi)}\left(\frac{k}{\sqrt{2}m}\right)^{2|\nu|-1} \tag{4.6}$$

for $\lambda = -k$.

The eigenfunctions so obtained are, by construction, automatically common eigenfunctions of H and Λ operators defined in the common domain given by the boundary conditions of equations (3.31) and (3.32), with the parameters satisfying equations (4.1)–(4.4). So the common eigenfunctions found can be explicitly written as

$$\psi_{|E|,\pm k}(kr) = \begin{pmatrix} J_{|\nu|}(kr) + c_+ J_{-|\nu|}(kr) \\ \frac{-\mathrm{i}k}{|E|+m}\left(J_{|\nu|-1}(kr) - c_+ J_{1-|\nu|}(kr)\right) \\ \frac{\pm k}{|E|+m}\left(J_{|\nu|}(kr) + d_+ J_{-|\nu|}(kr)\right) \\ \pm k\frac{\mathrm{i}}{k}\left(J_{|\nu|-1}(kr) - d_+ J_{1-|\nu|}(kr)\right) \end{pmatrix} \tag{4.7}$$

with $c_+ = d_+$ given by equation (4.5) for $\lambda = +k$ and with $c_+ = d_+$ given by equation (4.6) for $\lambda = -k$ and

$$\psi_{-|E|,\pm k}(kr) = \begin{pmatrix} J_{|\nu|}(kr) + c_- J_{-|\nu|}(kr) \\ \frac{+\mathrm{i}k}{|E|-m}\left(J_{|\nu|-1}(kr) - c_- J_{1-|\nu|}(kr)\right) \\ \frac{\pm k}{|E|-m}\left(J_{|\nu|}(kr) + d_- J_{-|\nu|}(kr)\right) \\ \pm k\frac{\mathrm{i}}{k}\left(J_{|\nu|-1}(kr) - d_- J_{1-|\nu|}(kr)\right) \end{pmatrix} \tag{4.8}$$

with $c_- = d_-$ given by equation (4.5) for $\lambda = +k$ and with $c_- = d_-$ given by equation (4.6) for $\lambda = -k$.

Substituting (4.1) into (4.4) in equations (3.31) and (3.32) and after some mathematical manipulations, the common domains can be characterized by

$$\begin{aligned} x &= m_\theta \sin\varphi w \\ y &= m_\theta \cos\varphi z \end{aligned} \tag{4.9}$$

where

$$m_\theta = \mathrm{i}\sqrt{2}(-)^{n_1}\cot\theta. \tag{4.10}$$

Note that this is the domain where H is self-adjoint.

Remark 4.1. As mentioned before H has a formal bound state. However the helicity operator does not even have formal bound states. Therefore the existence of a bound state for H would destroy commutativity. Using the asymptotic behaviour of the K_ν at the origin, it is easy to see that the boundary conditions (4.9) and (4.10) cannot be satisfied. Therefore H with the BCs (4.9) and (4.10) has no bound states.

There is no guarantee that Λ is self-adjoint in this domain. Therefore we have to answer the following two questions.

- Is the Λ operator self-adjoint in this domain?
- Does the imposition of self-adjointness for Λ lead to some extra relation between the parameters which are incompatible with equations (4.1)–(4.4)?

To answer these questions we can proceed as follows. First, notice that the admissible BCs for the helicity operator given by equations (3.21) and (3.22) can be written in terms of x, y, z and w, if we take $l = \sqrt{2}m$, as follows:

$$\begin{aligned}&(1+\overline{v_{11}}+\mathrm{i}\overline{v_{12}})w+(1-\overline{v_{11}}+\mathrm{i}\overline{v_{12}})z+(1+\overline{v_{11}}-\mathrm{i}\overline{v_{12}})y+(1-\overline{v_{11}}-\mathrm{i}\overline{v_{12}})x=0\\&(\mathrm{i}+\overline{v_{21}}+\mathrm{i}\overline{v_{22}})w+(-\mathrm{i}-\overline{v_{21}}+\mathrm{i}\overline{v_{22}})z+(-\mathrm{i}+\overline{v_{21}}-\mathrm{i}\overline{v_{22}})y+(\mathrm{i}-\overline{v_{21}}-\mathrm{i}\overline{v_{22}})x=0.\end{aligned}\tag{4.11}$$

This is the most general domain where helicity is a self-adjoint operator.

We impose that the two systems of equations (4.9) and (4.11) must have the same set of solutions. The necessary and sufficient condition for this to happen is that the following set of equations is satisfied:

$$\begin{aligned}&\overline{v_{11}}(1-m_\theta\sin\varphi)+\mathrm{i}\overline{v_{12}}(1-m_\theta\sin\varphi)=-(1+m_\theta\sin\varphi)\\&\overline{v_{11}}(1-m_\theta\cos\varphi)-\mathrm{i}\overline{v_{12}}(1-m_\theta\cos\varphi)=(1+m_\theta\cos\varphi)\end{aligned}\tag{4.12}$$

$$\begin{aligned}&\overline{v_{21}}(1-m_\theta\sin\varphi)+\mathrm{i}\overline{v_{22}}(1-m_\theta\sin\varphi)=-\mathrm{i}(1+m_\theta\sin\varphi)\\&\overline{v_{21}}(1-m_\theta\cos\varphi)-\mathrm{i}\overline{v_{22}}(1-m_\theta\cos\varphi)=-\mathrm{i}(1+m_\theta\cos\varphi).\end{aligned}\tag{4.13}$$

These decoupled systems always have solutions

$$v_{11}=v_{22}=\frac{m_\theta(\sin\varphi-\cos\varphi)}{(1+m_\theta\sin\varphi)(1+m_\theta\cos\varphi)}\tag{4.14}$$

$$v_{12}=-v_{21}=-\mathrm{i}\frac{(1-m_\theta^2\sin\varphi\cos\varphi)}{(1+\sin\varphi)(1+m_\theta\cos\varphi)}.\tag{4.15}$$

The imposition of unitarity for the matrix V thus obtained does not lead to any extra condition for the parameters. It is easy to see that the matrix V, given by equations (4.14) and (4.15), is automatically unitary.

Therefore, the Hamiltonian and the helicity are self-adjoint operators in the domains of equation (4.9) with the extension parameters satisfying equations (4.1)–(4.4). In this common domain it is possible to ensure commutativity and hence common eigenfunctions.

5. The Aharonov–Bohm symmetry

The class of BCs for H and Λ found in [1], as shown in the appendix, does not preserve helicity for all values of the parameters. Only for a particular BC ($\delta = 0$ or $\frac{\pi}{2}$), that is exactly the BC of [3, 4, 6], can we have helicity conservation, but in this case we violate the Aharonov–Bohm symmetry. This means that preservation of Aharonov–Bohm symmetry is incompatible with helicity conservation in the dynamics obtained in all the literature [1–4, 6].

In [3] it is shown that the BC of [4], although preserving helicity, leads to a breakdown of the Aharonov–Bohm symmetry (because of an asymmetry between $n > 0$ and $n < 0$ cases of the graph of the parameter θ of the extension as a function of the magnetic flux ϕ) and leads also to a discontinuity of the dynamics (i.e. of the BC and so of the wavefunctions) as a function of ϕ. It is also shown in [3] that the Aharonov–Bohm symmetry can be arranged by interpolation, but in this case we no longer have helicity conservation.

An interesting feature of the most general BCs given by equation (4.9), that include all the other BCs presented in the literature [1–4, 6], is that the simultaneous preservation of both symmetries can be arranged. Thus helicity conservation becomes compatible with Aharonov–Bohm symmetry for the class of dynamics we have obtained here.

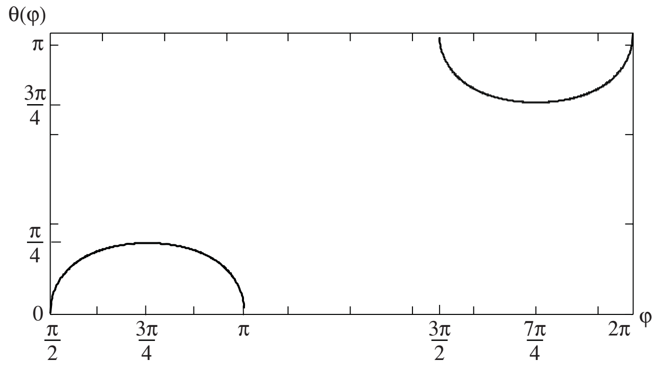

Figure 1. This graph shows the dependence between the parameters θ and φ that guarantees the helicity conservation when n_1 is even.

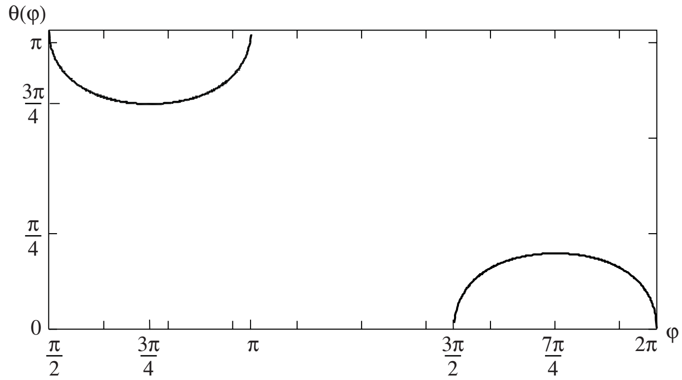

Figure 2. This graph shows the dependence between the parameters θ and φ that guarantees the helicity conservation when n_1 is odd.

To see how this occurs, we are going to analyse the continuity of the dynamics found in section 4 as a function of the flux ϕ. We are also going to analyse whether there is any asymmetry in the dynamics when the signal of the flux ϕ changes as occurs for the BCs of [3,4].

The domain defined by equation (4.9) with (4.3) satisfied is the most general domain where H and Λ effectively commute. (Notice that equation (4.4) only guarantees the denominator of equations (4.5) and (4.6) to be different from zero and so does not express specifically the condition of helicity conservation.) Notice also that equation (4.3) relates the parameters θ and φ and can be written as

$$\theta = \arccos\left\{(-)^{n_1}\frac{1}{\sqrt{2}}\operatorname{cosec}\left(\varphi - \frac{\pi}{4}\right)\right\}. \tag{5.1}$$

When n_1 is even, the equation above is defined in the intervals $\varphi : [\frac{\pi}{2}, \pi]$ with $\theta : [0, \frac{\pi}{4}]$ and $\varphi : [\frac{3\pi}{2}, 2\pi]$ with $\theta : [\frac{3\pi}{4}, \pi]$ (see figure 1).

When n_1 is odd, the equation (5.1) is defined in the intervals $\varphi : [\frac{\pi}{2}, \pi]$ with $\theta : [\frac{3\pi}{4}, \pi]$ and $\varphi : [\frac{3\pi}{2}, 2\pi]$ with $\theta : [0, \frac{\pi}{4}]$ (see figure 2).

For the situation $\nu \geqslant 0$ (i.e. $\phi \geqslant -n$, for any fixed value of the angular momentum n), the only admissible BC is the regular solution at the origin (otherwise the wavefunctions are not square integrable in the neighbourhood of the origin).

For one to have only the regular solution (for $\nu > 0$) in equations (2.12) and (2.13), the BCs of equation (4.9) must be given by

$$\begin{aligned} &\theta = 0,\ 2\pi \qquad \text{with} \quad \varphi = \frac{\pi}{2} \\ \text{or} \\ &\theta = \pi \qquad \text{with} \quad \varphi = \frac{3\pi}{2} \end{aligned} \tag{5.2}$$

when n_1 is even and

$$\begin{aligned} &\theta = 0,\ 2\pi \qquad \text{with} \quad \varphi = \frac{3\pi}{2} \\ \text{or} \\ &\theta = \pi \qquad \text{with} \quad \varphi = \frac{\pi}{2} \end{aligned} \tag{5.3}$$

when n_1 is odd. The parameters above were taken to satisfy the condition (5.1) of helicity conservation.

For the case $\nu \leqslant -1$ (i.e. $\phi \leqslant -n - 1$ for any fixed value of the angular momentum n), again only the regular solutions at the origin are admissible.

In order to have only regular solutions in equations (2.12) and (2.13) (for $\nu \leqslant -1$), the BCs of equation (4.9) must be given by

$$\begin{aligned} &\theta = 0,\ 2\pi \qquad \text{with} \quad \varphi = \pi \\ \text{or} \\ &\theta = \pi \qquad \text{with} \quad \varphi = 0, 2\pi \end{aligned} \tag{5.4}$$

when n_1 is even and

$$\begin{aligned} &\theta = 0,\ 2\pi \qquad \text{with} \quad \varphi = 0, 2\pi \\ &\theta = \pi \qquad \text{with} \quad \varphi = \pi \end{aligned} \tag{5.5}$$

when n_1 is odd. The parameters above were taken again to satisfy condition (5.1), so these BCs also preserve helicity.

From these results, we see that for certain choices of the parameters it is possible to obtain, by interpolation, the continuity of the dynamics as a function of the magnetic flux ϕ and also preservation of Aharonov–Bohm symmetry.

In order to ensure continuity of the wavefunctions as a function of ϕ for a given value of the angular momentum n, and also for n_1 even, we can interpolate, for example, the chosen functions

$$\begin{aligned} &\varphi = \tfrac{\pi}{2} \qquad \text{with} \quad \phi \geqslant -n \\ &\varphi = \pi \qquad \text{with} \quad \phi \leqslant -n - 1 \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} &\theta(\phi) = 0 \qquad \phi \geqslant -n \\ &\theta(\phi) = 0 \qquad \phi \leqslant -n - 1 \end{aligned} \tag{5.7}$$

in the open interval $-n - 1 < \phi < -n$ (i.e. $\nu : [-1, 0]$).

Notice that interpolation with helicity conservation (i.e. with condition (5.1) satisfied) is possible for these intervals of variation of the parameters, if we choose

$$\varphi(\phi) = f(\phi + n + 1) \qquad -n - 1 < \phi < -n \tag{5.8}$$

where f is a fixed n-independent continuous function defined in the interval (0, 1) and such that $f(0) = 0$ and $f(1) = \frac{\pi}{2}$.

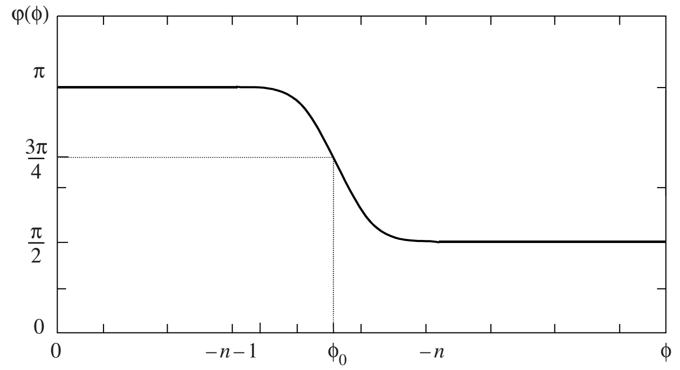

Figure 3. Interpolation with helicity conservation for the parameter φ as a function of the magnetic flux ϕ (for negative angular momentum n) when n_1 is even.

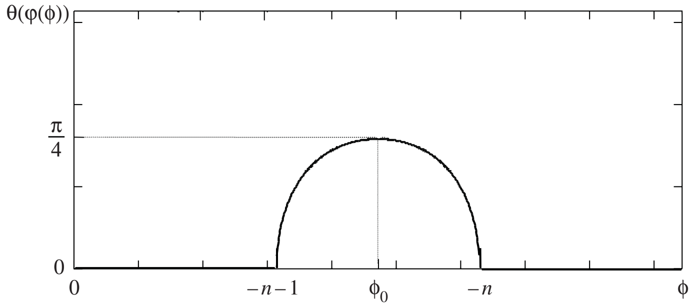

Figure 4. Interpolation with helicity conservation for the parameter θ as a function of the magnetic flux ϕ (for negative angular momentum n) when n_1 is even.

It is crucial that, for the chosen interval of variation of φ, it is possible to have condition (5.1) of helicity conservation satisfied if we take

$$\theta(\varphi(\phi)) = \arccos \frac{1}{\sqrt{2}} \operatorname{cosec}\left(f(\phi + n + 1) - \frac{\pi}{4}\right) \tag{5.9}$$

with $\theta(\varphi(\phi))$ in the interval $[0, \frac{\pi}{4}]$ (see figures 2–4).

Notice also that, for $n > 0$ (positive angular momentum), the graphics of figures 3 and 4 will have the same aspect. So there is no asymmetry between the $n > 0$ and $n < 0$ cases in contrast to the BC considered in [3, 4].

This is not the case of the BCs discussed in [3, 4], which were obtained by the limiting procedure of a magnetic flux tube of ratio $R > 0$.

Another possible choice of the parameters for n_1 even is shown in figures 5 and 6.

For n_1 odd, it is possible to guarantee the preservation of the $\phi \to \phi + 1$ symmetry and continuity of the dynamics as a function of ϕ, with condition (5.1) also satisfied, if we interpolate the chosen functions

$$\begin{aligned} \theta(\phi) &= 0 \qquad \phi \geqslant -n \\ \theta(\phi) &= 0 \qquad \phi \leqslant -n - 1 \end{aligned} \tag{5.10}$$

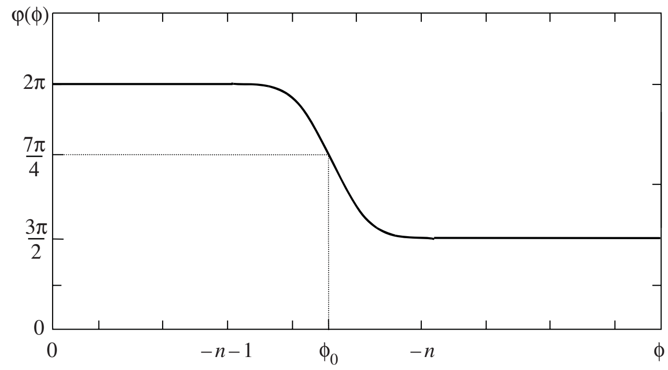

Figure 5. Interpolation with helicity conservation for the parameter φ as a function of the magnetic flux ϕ (for negative angular momentum n) when n_1 is even.

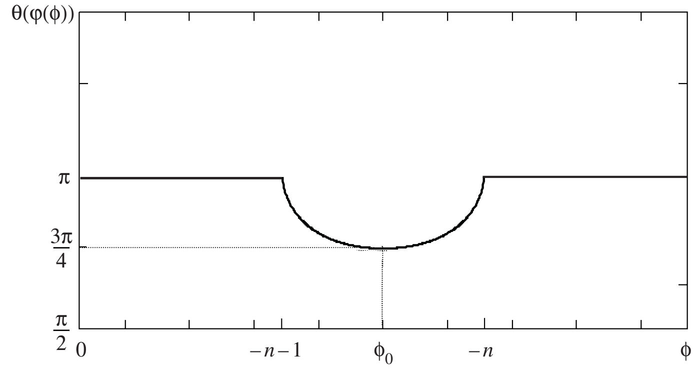

Figure 6. Interpolation with helicity conservation for the parameter θ as a function of the magnetic flux ϕ (for negative angular momentum n) when n_1 is even.

and

$$\begin{aligned} \varphi &= \frac{3\pi}{2} \qquad \phi \geqslant -n \\ \varphi &= 2\pi \qquad \phi \leqslant -n-1 \end{aligned} \tag{5.11}$$

in the open interval $-n-1 < \phi < -n$.

We choose

$$\varphi(\phi) = g(\phi + n + 1) \qquad -n-1 < \phi < -n$$

where g is a fixed n-independent continuous function defined in the interval (0, 1) and such that $g(0) = 2\pi$ and $g(1) = \frac{3\pi}{2}$.

Again it is crucial that for this interval of variation of φ it is possible to have the helicity conservation condition (5.1) satisfied if we take

$$\theta(\varphi(\phi)) = \arccos\left\{-\frac{1}{\sqrt{2}}\operatorname{cosec}\left(g(\phi + n + 1) - \frac{\pi}{4}\right)\right\} \tag{5.12}$$

with $\theta(\varphi(\phi))$ in the interval $[0, \frac{\pi}{4}]$ (see figures 2, 7 and 8).

Another possible choice of the parameters when n_1 odd is shown in figures 9 and 10.

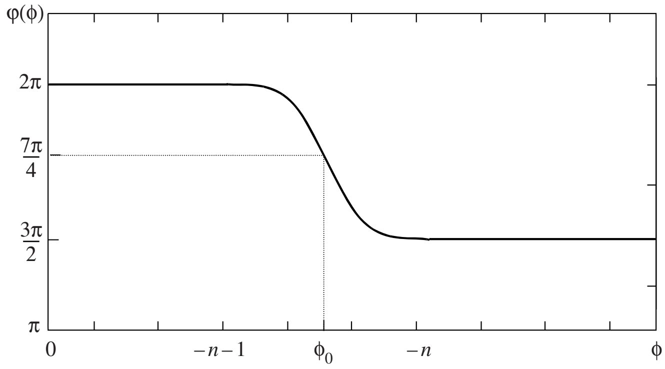

Figure 7. Interpolation with helicity conservation for the parameter φ as a function of the magnetic flux ϕ (for negative angular momentum n) when n_1 is odd.

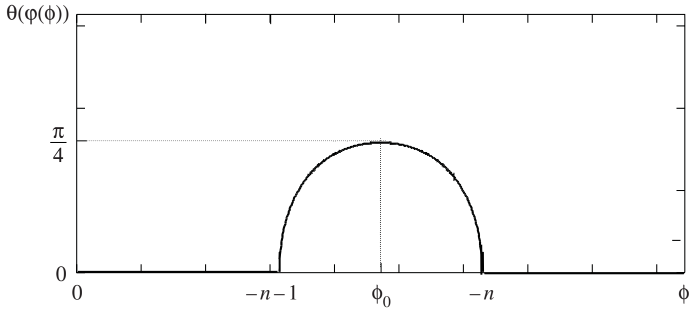

Figure 8. Interpolation with helicity conservation for the parameter θ as a function of the magnetic flux ϕ (for negative angular momentum n) when n_1 is odd.

Notice that again the graphics of figures 7–10 do not change their aspect when the angular momentum n is positive. So there is no asymmetry between the $n > 0$ and $n < 0$ cases.

Therefore, the BCs of section 4, that ensure helicity conservation, can also preserve the $\phi \to \phi + 1$ symmetry.

The continuity of the dynamics (and of the wavefunctions) as a function of the flux ϕ is also obtained.

6. Conclusions

It is important to say a few words about the interesting features of the BCs we have obtained here.

First, the BCs of equations (3.31) and (3.32) characterize the most general domain where the Hamiltonian is a self-adjoint operator and include all the other BCs presented in the literature [1–4, 6].

Helicity conservation can be imposed by the satisfaction of the formal condition (2.22) that leads to certain relations between the parameters of the extensions given by (4.1)–(4.4). This defines new domains given by (4.9) with the parameters obeying (4.3).

Besides helicity conservation, preservation of the Aharonov–Bohm symmetry can be arranged by interpolation for certain choices of the parameters. The continuity of the dynamics

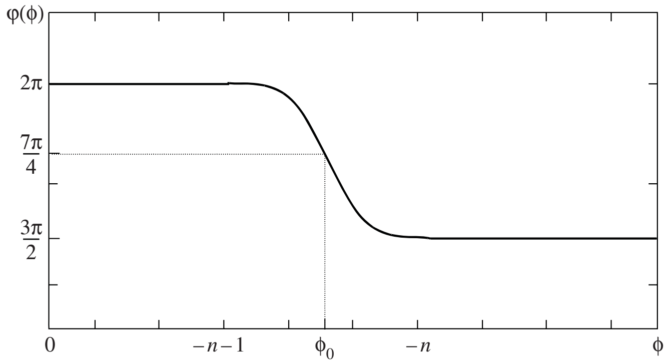

Figure 9. Interpolation with helicity conservation for the parameter φ as a function of the magnetic flux ϕ (for negative angular momentum n) when n_1 is even.

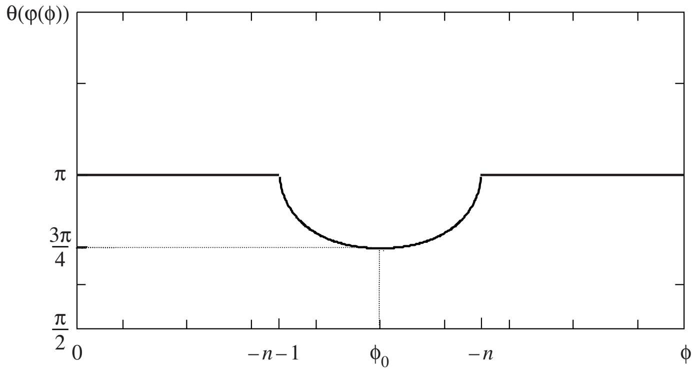

Figure 10. Interpolation with helicity conservation for the parameter θ as a function of the magnetic flux ϕ (for negative angular momentum n) when n_1 is odd.

as a function of the magnetic flux ϕ is also guaranteed. Thus helicity conservation becomes compatible with Aharonov–Bohm symmetry and with the continuity of the dynamic for the class of BCs we have obtained here.

Acknowledgments

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Appendix

We have seen in this paper that surprisingly helicity conservation is not obtained by only taking the same BCs for H and Λ, as claimed in [1]. There is a formal condition that implies some relations between the parameters that must be satisfied.

The BCs discussed in [1] are a particular case of the most general BCs given by equations (3.31) and (3.32) of this paper. We can obtain the W matrix given by equation (A.9) of [1] from our equations (3.31) and (3.32) by considering the following relations between the parameters.

$$\theta = 0 \tag{A.1}$$

$$\alpha + \varphi = w_1 \tag{A.2}$$

$$\alpha - \varphi = w_2. \tag{A.3}$$

The parameters w_1 and w_2, as shown in [1], are related to the parameters δ_1 and δ_2 by

$$\tan\delta_i = \frac{\sqrt{2}}{\tan\frac{w_i}{2} - 1} \qquad i = 1, 2 \tag{A.4}$$

where (δ_1, δ_2) describe the special class of BCs

$$\begin{aligned} x\cos\delta_1 - \mathrm{i}y\sin\delta_1 &= 0 \\ z\cos\delta_2 - \mathrm{i}w\sin\delta_2 &= 0. \end{aligned} \tag{A.5}$$

The BCs above have the special feature of preserving the original symmetry of the Hamiltonian that allowed it to be represented by two independent operators acting on two-component spinors.

When we impose self-adjointness of the helicity operator defined in the domain given by (A.5), we obtain the condition

$$\delta_1 - \delta_2 = 2\pi k \qquad k \in Z. \tag{A.6}$$

Because of the periodicity of BCs given by (A.5), the condition above reads

$$\delta_1 = \delta_2 = \delta. \tag{A.7}$$

Imposing the most general eigenfunction of H given by (2.14), (2.12) and (2.13) to be defined in the domain of (A.5) with condition (A.7) satisfied, we obtain, for a fixed energy value

$$c_\pm = \frac{\tan\delta}{\pm|E| + m}\frac{k^{2|\nu|}}{(\sqrt{2}m)^{2|\nu|-1}} \tag{A.8}$$

and

$$d_\pm = \frac{\tan\delta}{\pm|E| - m}\frac{k^{2|\nu|}}{(\sqrt{2}m)^{2|\nu|-1}}. \tag{A.9}$$

Common eigenfunctions for H and Λ, as we have seen in section 2, are possible for some particular linear combinations given by (2.26) and (2.27), but this requires also that equation (2.22) be satisfied. The condition (2.22) is only verified for the particular values of the extensions' parameters

$$\delta = 0 \tag{A.10}$$

and

$$\delta = \frac{\pi}{2}. \tag{A.11}$$

The $\delta = 0$ solution implies $c_\pm = d_\pm = 0$, and so corresponds to a regular upper component and singular lower component of the spinor of equations (2.12) and (2.13). The $\delta = \frac{\pi}{2}$ solution implies $c_\pm = d_\pm = \infty$, and so corresponds to a singular upper component and regular lower component of the spinor of equations (2.12) and (2.13).

This result is the same as found in [3] and [4], obtained by the limiting procedure of the magnetic flux tube of ratio $R > 0$ [3] and [4]. It is also the same result obtained in [6], considering the case $s = +1$. (For $s = -1$ in [6], one can find a correspondence between the results, considering that there is a unitary correspondence between the Hamiltonian operator used in this paper and also in [1–4] and the Hamiltonian used in [6].)

So only for these special SAs ,and not for all SAs given by (A.5), is it possible to have common eigenfunctions for the self-adjoint operators H and Λ. For these special SAs a breakdown of the Aharonov–Bohm symmetry appears [3, 4].

This result disagrees with [1] and it is unexpected. It seems to occur because both operators H and Λ, when acting in the domain defined by (A.5), do not leave it invariant.

Let us consider an eigenfunction of H given by (2.14) defined in the domain given by (A.5).

Acting with the helicity operator, given by (2.15), on this eigenfunction and using equations (2.7) and (2.8), we obtain

$$\Phi_{\pm} = \Lambda\psi_{\pm} = \pm|E| + m \begin{pmatrix} \beta\left(J_{|\nu|}(kr) + d_{\pm}J_{-|\nu|}(kr)\right) \\ \frac{-\mathrm{i}k}{\pm|E|-m}\beta\left(J_{|\nu|-1}(kr) - d_{\pm}J_{-|\nu|+1}(kr)\right) \\ \frac{\pm|E|-m}{\pm|E|+m}\alpha\left(J_{|\nu|}(kr) + c_{\pm}J_{-|\nu|}(kr)\right) \\ \frac{-\mathrm{i}k}{\pm|E|+m}\left(J_{|\nu|-1}(kr) - c_{\pm}J_{-|\nu|+1}(kr)\right) \end{pmatrix}. \tag{A.12}$$

As we can immediately see, the wavefunction $\Phi_{\pm}$ above no longer satisfies the BCs given by (A.5) and so it is not in the common domain of H and Λ, where both operators are self-adjoint.

In an analogous way, one can easily verify that the Hamiltonian operator, when acting in the common domain given by (A.5), generates wavefunctions of H in the domain of equation (A.5) which no longer satisfy these BCs.

This seems to be a physical example of the Nelson problem presented in [7].

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